Initial Orbit Determination from Atmospheric Drag Direction

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Nomenclature

\( a_b \) = specific force acceleration in satellite body frame, m/s\(^2\)

\( a_{drag} \) = drag acceleration, m/s\(^2\)

\( a_s \) = solar radiation pressure acceleration, m/s\(^2\)

\( a \) = semi-major axis, m

\( C_{b}^{i} \) = coordinate transformation matrix from satellite body frame to Earth-centered inertial frame

\( C_d \) = drag coefficient

\( f \) = true anomaly, rad

\( h_p \) = perigee altitude

\( m \) = mass of satellite, kg

\( P_x \) = covariance matrix of initial orbit determination error

\( R \) = covariance matrix of normalized drag acceleration measurement error

\( r \) = satellite position vector in Earth-centered inertial frame, m

\( S \) = projected area vertical to direction of satellite’s velocity relative to atmosphere, m\(^2\)

\( T \) = orbital period, s

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\[ u_{di} = \text{normalized drag acceleration in Earth-centered inertial frame} \]
\[ u_i = \text{normalized specific force acceleration in Earth-centered inertial frame} \]
\[ v = \text{satellite absolute velocity in Earth-centered inertial frame, m/s} \]
\[ v_a = \text{local atmosphere velocity in Earth-centered inertial frame, m/s} \]
\[ v_r = \text{satellite velocity relative to the atmosphere, m/s} \]
\[ \varepsilon = \text{measurement noise of spaceborne accelerometer, m/s}^2 \]
\[ \theta = \text{attitude Euler angle, rad} \]
\[ \mu = \text{gravitational constant of Earth, m}^3/\text{s}^2 \]
\[ \rho = \text{local density of Earth atmosphere, kg/m}^3 \]
\[ \omega_e = \text{Earth rotational angular velocity vector, rad/s} \]

I. Introduction

The motion of low Earth orbiting (LEO) satellites, especially those orbiting at very low altitudes, are influenced by the Earth’s atmosphere. The drag accounts for most of the forces exerted by the atmosphere on the satellite. Its direction opposes the satellite’s velocity relative to the atmosphere, and its magnitude is proportional to the square of the relative velocity. Considering large uncertainties in empirical atmospheric density models and ballistic coefficient, the magnitude of atmospheric drag usually cannot be accurately modeled. In contrast, modeling of drag direction is subject to less uncertainty since the wind speed is much smaller than the satellite’s velocity and accurate wind models such as the new Horizontal Wind Model (HWM07) have been developed nowadays [1]. The atmospheric drag can be sensed by a high-precision electrostatic or superconducting accelerometer if the other dominating non-conservative forces, for example, the solar radiation pressure could be removed from measurements by precise modeling [2,3]. Then the drag direction can be utilized to derive the satellite’s velocity information and to further determine the initial orbit. Compared to orbit determination using ground-tracking measurements or Global Navigation Satellite System (GNSS) signals, this approach is fully autonomous and passive, which provides a means of covert navigation or can be used as backup for other autonomous onboard navigation systems.

The measurement of non-conservative accelerations of LEO spacecraft requires high sensitivity of spaceborne accelerometers. During the last few decades, electrostatic and superconducting accelerometers have been proposed
and designed for space applications involving accelerometry with high precision. The French national aerospace research center (ONERA) developed a series of three-axis electrostatic accelerometers which have been successfully implemented on recent satellite gravimetry missions including CHAllenging Minisatellite Payload (CHAMP), Gravity Recovery and Climate Experiment (GRACE), Gravity field and steady-state Ocean Circulation Explorer (GOCE), and GRACE Follow-On [4–7]. Taking the electrostatic accelerometers on GOCE for example, a noise density level of $10^{-12} \text{m/s}^2/\sqrt{\text{Hz}}$ was reached within the frequency band of 5–100 mHz [6]. Researchers from the University of Maryland have long been engaged in superconducting accelerometry as well as its applications since 1980s [8]. They are now developing a spaceborne gravity gradiometer consisting of superconducting accelerometers with a sensitivity of $1.4 \times 10^{-13} \text{m/s}^2/\sqrt{\text{Hz}}$ in the frequency band of 1–50 mHz for future Mars gravity recovery [9]. The similar superconducting accelerometer technology is also expected to be employed in the recently proposed Satellite Test of the Equivalence Principle (STEP) mission to test the equivalence principle of general relativity [10].

This Note investigates the feasibility of using a high-precision spaceborne accelerometer for orbit determination of LEO satellites in low altitude regime where the atmospheric drag is prominent. The non-conservative accelerations are measured in the satellite’s body frame and are transformed to the Earth-centered inertial (ECI) frame via star sensor measurements. The drag accelerations can be isolated from the total non-conservative accelerations by modeling of solar radiation pressure. The drag direction which opposes the satellite’s velocity relative to the wind is then utilized for initial orbit determination. It should be noted that theoretical feasibility analysis rather than practical system design is the major concern of this study. Thus technical considerations such as accelerometer design and maintenance are not discussed. In addition, the bandwidth limitation of electrostatic and/or superconducting accelerometers induces measurement biases. Thanks to the high performance of these accelerometers, the biases are usually stable and can be calibrated in advance. Furthermore, this study should be distinguished from a recent proposed problem of initial orbit determination using three velocity vectors [11]. It is the direction of the satellite’s velocity relative to the atmosphere, instead of the satellite’s absolute velocity or its direction, that is used for orbit determination in this study.

The remainder of this Note is organized as follows. Section II describes the principle of LEO orbit determination using spaceborne accelerometer and presents the problem of initial orbit determination from drag direction. The algorithm design and error covariance analysis are given in Section III and IV, respectively. Section V presents the results of numerical simulations, and the conclusion is drawn in Section VI.
II. Orbit Determination Scheme

Consider a satellite in LEO. The atmospheric drag can be expressed as follows

\[
a_{\text{drag}} = -\frac{1}{2} C_d \frac{S}{m} \rho \left| \mathbf{v}_r \right| \mathbf{v}_r
\]  

(1)

where \(a_{\text{drag}}\) represents the drag acceleration vector, \(C_d\) is the drag coefficient, \(\frac{S}{m}\) is the area-to-mass ratio, \(\rho\) is the local density of the atmosphere, \(\mathbf{v}_r\) is the satellite velocity relative to the atmosphere and \(\left| \mathbf{v}_r \right|\) is the magnitude. The relative velocity can be expressed as

\[
\mathbf{v}_r = \mathbf{v} - \mathbf{v}_a
\]  

(2)

where \(\mathbf{v}\) is the satellite’s absolute velocity, \(\mathbf{v}_a\) is the local atmosphere velocity. Several models can be utilized to calculate \(\mathbf{v}_a\) accurately such as HWM07. For a feasibility study, it can be assumed that the Earth atmosphere co-rotates with the Earth surface. Then it can be expressed as

\[
\mathbf{v}_a = \omega_e \times \mathbf{r}
\]  

(3)

where \(\omega_e\) is the Earth rotational angular velocity vector, \(\mathbf{r}\) is the satellite’s position vector in the ECI frame.

Fig. 1 depicts the magnitude of drag acceleration for a simplified model of satellites in circular orbits with altitudes ranging from 200 km to 1000 km. The area-to-mass ratio is set to 0.02 m²/kg and the drag coefficient is set to 2.2. In this Note, the orbital altitude range of 200–400 km is adopted, since in this altitude range the magnitude of atmospheric drag acceleration is large enough for accurate initial orbit determination.

![Drag acceleration of satellites in circular orbits with respect to orbital altitude.](image)

Fig. 1 Drag acceleration of satellites in circular orbits with respect to orbital altitude.

Now consider a three-axis ultra-sensitive accelerometer installed at the mass center of a LEO satellite whose axes are aligned to those of the satellite’s body frame. Then the non-conservative force acceleration without considering measurement error can be expressed as follows
\[
a_i = C_{ib} \left( a_{di} + a_{si} \right)
\]

where \( a_{di} \) and \( a_{si} \) represent drag acceleration and solar radiation pressure acceleration in ECI, respectively, and \( C_{ib} \) is the coordinate transformation matrix from ECI to the satellite’s body frame.

By using spaceborne star sensor, the coordinate transformation matrix \( C_{ib} \) can be obtained and the non-conservative acceleration measurement can be transformed to the ECI frame as follows

\[
a_i = C_{ib} a_b = a_{di} + a_{si}
\]

Then the drag acceleration can be obtained by removing the solar radiation pressure acceleration from \( a_i \)

\[
a_{di} = a_i - a_{si}
\]

where \( a_{si} \) can be modeled and obtained accurately.

As stated earlier, large uncertainties exist in empirical atmospheric density models and ballistic coefficient, thus the drag acceleration direction instead of its magnitude is utilized to derive the orbit information. The drag acceleration direction (negative direction) can be obtained by

\[
\mathbf{u}_{di} = -\frac{a_{di}}{|a_{di}|}
\]

It has a relationship with the satellite’s velocity

\[
\mathbf{u}_{di} = \frac{\mathbf{v}_r}{|\mathbf{v}_r|} = \frac{\mathbf{v} - \omega_E \times \mathbf{r}}{|\mathbf{v} - \omega_E \times \mathbf{r}|}
\]

Intuitively, \( \mathbf{u}_{di} \) at three times is enough for initial orbit determination. However, there are only five independent quantities in three observations of \( \mathbf{u}_{di} \). The proof is given as follows.

Construct a new variable

\[
p = \omega_E \times \mathbf{u}_{di}
\]

Then

\[
p_0 = \omega_E \times \frac{\mathbf{v}_0 - \omega_E \times \mathbf{r}_0}{|\mathbf{v}_0 - \omega_E \times \mathbf{r}_0|} = \omega_E \times \frac{\mathbf{v}_0}{|\mathbf{v}_0 - \omega_E \times \mathbf{r}_0|}
\]

\[
p_1 = \omega_E \times \frac{\mathbf{v}_1 - \omega_E \times \mathbf{r}_1}{|\mathbf{v}_1 - \omega_E \times \mathbf{r}_1|} = \omega_E \times \frac{\mathbf{v}_1}{|\mathbf{v}_1 - \omega_E \times \mathbf{r}_1|}
\]

\[
p_2 = \omega_E \times \frac{\mathbf{v}_2 - \omega_E \times \mathbf{r}_2}{|\mathbf{v}_2 - \omega_E \times \mathbf{r}_2|} = \omega_E \times \frac{\mathbf{v}_2}{|\mathbf{v}_2 - \omega_E \times \mathbf{r}_2|}
\]
It can be easily seen that

\[
(p_0 \times p_1) \cdot p_2 = 0
\]  

That is to say if two observations already exist, one more observation brings about only one more independent quantity. Hence four observations can provide six independent quantities totally which are enough for initial orbit determination. It is worth noting that there is no multiple solution problem when solving the initial orbit determination problem utilizing atmospheric drag directions, because it is easy for LEO satellite to determine whether the interval between the two drag measurements is less than one orbital revolution or more than one orbital revolution. Fig. 2 shows a simplified block diagram of this initial orbit determination scheme.

**Fig. 2** Block diagram of the initial orbit determination scheme.

## III. Algorithm Design

### A. Initial Value Formulas

In order to solve the initial orbit determination equations numerically utilizing Newton iteration method, a set of simple initial value formulas are proposed below. Assume that the atmospheric drag acceleration dominates in all kinds of non-conservative force accelerations. Firstly, transform the first two non-conservative acceleration measurement values to the ECI frame with the help of the first two coordinate transformation matrices. Then normalize the first two non-conservative acceleration measurement values in ECI as follows

\[
\begin{bmatrix}
  u_{i0} \\
  u_{i1}
\end{bmatrix} = \begin{bmatrix}
  C_{00}^i a_{00} \\
  C_{01}^i a_{01}
\end{bmatrix},
\]

\[
\begin{bmatrix}
  u_{i1} \\
  u_{i2}
\end{bmatrix} = \begin{bmatrix}
  C_{10}^i a_{10} \\
  C_{11}^i a_{11}
\end{bmatrix}
\]


Since the proposed initial orbit determination method is only applicable to the LEO orbits which are approximate to circular orbits, circular orbit assumption is adopted in this part in order to simplify the calculation. Then the angle between position vectors of two sampling times is

\[
\Delta f = \begin{cases} 
\arccos(\mathbf{u}_0 \cdot \mathbf{u}_1), & \Delta t < \frac{1}{2}T \\
2\pi - \arccos(\mathbf{u}_0 \cdot \mathbf{u}_1), & \frac{1}{2}T < \Delta t < T 
\end{cases}
\] (13)

where \(T\) represents the orbital period.

Then the semi-major axis is

\[
a = \left( \frac{\mu \Delta t^2}{\Delta f^2} \right)^{\frac{1}{3}}
\] (14)

where \(\mu = GM\) is gravitational constant of the Earth, \(G\) is gravitational constant, and \(M\) is the mass of the earth.

Then the magnitude of velocity of the satellite can be

\[
v_{\text{circular}} = \sqrt{\frac{\mu}{a}}
\] (15)

Hence the initial value of velocity vector is

\[
v_0 = v_{\text{circular}} u_{0i}
\] (16)

In order to obtain the initial value of position vector, the positive or negative product of \(u_{0i}\) and \(u_{1i}\) can be expressed as

\[
\hat{\hat{H}} = \begin{cases} 
\mathbf{u}_{0i} \times \mathbf{u}_{1i}, & \Delta t < \frac{1}{2}T \\
-\mathbf{u}_{0i} \times \mathbf{u}_{1i}, & \frac{1}{2}T < \Delta t < T 
\end{cases}
\] (17)

then the initial value of position vector is

\[
r_0 = \frac{\mathbf{u}_{0i} \times \hat{\hat{H}}}{|\mathbf{u}_{0i} \times \hat{\hat{H}}|} a
\] (18)

So far, the formulas of initial values for numerical solution has been provided.

B. Initial Orbit Determination Solution
The proposed initial orbit determination equations can be solved by Newton iteration method. Firstly, the measurement values are rewritten as a column vector as

\[
y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\
\end{bmatrix}
\]

(19)

And the position and velocity of the satellite which are to be solved are rewritten as a column vector as

\[
x = \begin{bmatrix} r_0 \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ \end{bmatrix}
\]

(20)

Then the iterative solution formula can be written as follows

\[
x^{j+1} = x^j + \left( H^T R^{-1} H \right)^{-1} H^T R^{-1} (y - \bar{y}), \quad j = 0, 1, 2, \ldots
\]

(21)

where

\[
\bar{y} = \begin{bmatrix} v_0 - \omega_0 \times r_0 \\ v_0 - \omega_0 \times r_1 \\ v_1 - \omega_0 \times r_1 \\ v_1 - \omega_0 \times r_2 \\ v_2 - \omega_0 \times r_2 \\ v_2 - \omega_0 \times r_3 \\ v_3 - \omega_0 \times r_3 \\ \end{bmatrix}
\]

(22)

represents the normalized drag acceleration vector which is obtained by Earth rotation modeling. \( j \) represents the number of iterations. \( R \) is the covariance matrix of the errors of normalized drag acceleration measurements. \( \left( H^T R^{-1} H \right)^{-1} \) represents the weighted pseudo inverse of gradient matrix \( H \) which is expressed as

\[
H = \begin{bmatrix} M_0 N_0 \\ M_1 N_1 \\ M_2 N_2 \\ M_3 N_3 \\ \end{bmatrix}
\]

(23)

where

\[
M_k = \begin{bmatrix} \frac{\partial y_k}{\partial r_k} \\ \frac{\partial y_k}{\partial v_k} \\ \end{bmatrix}, \quad k = 0, 1, 2, 3
\]

(24)
\[ N_k = \begin{bmatrix} \frac{\partial r_i}{\partial r_0} & \frac{\partial r_i}{\partial v_0} \\ \frac{\partial v_i}{\partial r_0} & \frac{\partial v_i}{\partial v_0} \end{bmatrix}, \quad k = 0, 1, 2, 3 \]

(25)

Take the derivative of \( M_i \) with respect to position and velocity

\[ \frac{\partial y_i}{\partial r_i} = \frac{1}{|v_i - \omega \times r_i|} \left( \omega_j \right)^T + \frac{1}{|v_i - \omega \times r_i|} \left( v_i - \omega \times r_i \right) \left( \omega_j \right)^T + r_i^T \left( \omega_j \right)^T \]

\[ \frac{\partial y_i}{\partial v_i} = \frac{1}{|v_i - \omega \times r_i|} I_{3 \times 3} \left( v_i - \omega \times r_i \right) \left( v_i \right)^T + r_i^T \left( \omega_j \right)^T \]

(26)

The elements in \( N_k \) can be expressed as follows according to Battin’s book [12]

\[ \frac{\partial r}{\partial v_0} = \frac{r}{\mu} (1 - F) \left( \left( r - r_0 \right) v_0^T - (v - v_0) r_0^T \right) + \frac{C}{\mu} v v_0^T + GI \]

\[ \frac{\partial v}{\partial v_0} = \frac{r}{\mu} \left( v - v_0 \right) (v - v_0)^T + \frac{1}{r^3} \left( r_0 \left( 1 - F \right) r_0^T - Cr v_0^T \right) + G I \]

\[ \frac{\partial r}{\partial r_0} = \frac{r}{\mu} \left( v - v_0 \right) r_0^T + \frac{1}{r^3} \left( r_0 \left( 1 - F \right) r_0^T + C v r_0^T \right) + FI \]

\[ \frac{\partial v}{\partial r_0} = -\frac{1}{r^3} \left( v - v_0 \right) r_0^T - \frac{1}{r^3} r \left( v - v_0 \right) r_0^T + F \left[ I - \frac{1}{r^3} r r^T + \frac{1}{\mu r} \left( r v^T - v r^T \right) r \left( v - v_0 \right) \right] - \frac{\mu C}{r^3} r_0 r_0^T \]

(27)

where \( F, G, F_i, G_i \) are Lagrange coefficients described by universal variable function \( U_\nu (\chi, \alpha) \)

So far, the solution to the initial orbit determination has been formulated completely.

C. Overall Design of the Algorithm

<table>
<thead>
<tr>
<th>Step 1:</th>
<th>Obtain acceleration measurements at four times ( {a_0, a_1, a_2, a_3} ) from spaceborne accelerometer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 2:</td>
<td>Obtain attitude information of four times ( {C_{b0}, C_{b1}, C_{b2}, C_{b3}} ) from star sensor and transform the acceleration measurements to the ECI frame using Eq. (5).</td>
</tr>
<tr>
<td>Step 3:</td>
<td>Obtain initial values of position and velocity using Eqs. (13)–(18).</td>
</tr>
<tr>
<td>Step 4:</td>
<td>Establish and isolate the initial orbit determination equations [Eq. (23)] with Newton iteration method. Position and velocity values are updated.</td>
</tr>
<tr>
<td>Step 5:</td>
<td>The position and velocity have been found, and the initial orbit determination problem is complete.</td>
</tr>
</tbody>
</table>
Considering the process of obtaining solar radiation pressure acceleration relies on the position and velocity of the satellite, a two-layer iteration algorithm is designed. The outer iteration is used for the coordinate transformation of the measurements and the elimination of the solar radiation pressure acceleration, and the inner iteration is used for solving the initial orbit determination equations numerically. The overall design of this algorithm is summarized in Table. 1.

IV. Error Covariance Analysis

In the measurement and calculation process of the proposed initial orbit determination, different sources of error, including acceleration measurement noise, star sensor measurement noise, solar radiation pressure model error and atmosphere rotational velocity error, have influence on the orbit determination result. In this section, the transfer process of different sources of error to the orbit determination result is analyzed.

For the spaceborne accelerometer, a real measurement can be expressed as

\[
z = C^b_i \left[ -\lambda \left( v - (\omega \times r + \Delta v_s) \right) + a_s \right] + \epsilon
\]

where \( z \) represents real non-conservative force acceleration measurement, \( C^b_i \) represents the true coordinate transformation matrix from the ECI frame to satellite body frame. The relevant coefficients are rewritten as

\[
\lambda = \frac{1}{2} \rho C_a \frac{S}{m} |v_s| \quad \Delta v_s \quad \text{represents the earth atmosphere rotational velocity error.}
\]

\( a_s \) represents true acceleration of satellite due to solar radiation pressure. \( \epsilon \) is the measurement noise of the spaceborne accelerometer. The biases of accelerometer are not considered in this Note.

After the coordinate transformation and solar radiation pressure elimination, the drag acceleration vector in the ECI frame can be obtained as follows

\[
\tilde{C}_b \tilde{z} - \tilde{a}_s = -\lambda \left( v - (\omega \times r) \right) + \tilde{C}_b \Delta C \lambda \Delta v_s - \tilde{C}_b \Delta C \Delta a_s + \tilde{C}_b \Delta C \lambda \left( v - (\omega \times r) \right) - \tilde{C}_b \Delta C \lambda \Delta a_s + \lambda \Delta v_s + \Delta a_s + \tilde{C}_b \epsilon
\]  

where \( \tilde{C}_b \) represents the coordinate transformation matrix from satellite body frame to the ECI frame obtained from the measurement of star sensor. \( \tilde{a}_s \) represents the solar radiation pressure acceleration in the ECI frame which is obtained from solar radiation pressure model. \( \Delta C^b = \tilde{C}_b - C^b \) is the coordinate transformation matrix error and \( \Delta a_s = a_s - \tilde{a}_s \) is the modeling error of solar radiation pressure. The second and third item on the right side of Eq. (29), \( \tilde{C}_b \Delta C \lambda \Delta v_s \) and \( -\tilde{C}_b \Delta C \Delta a_s \), can be ignored since they are high order small terms.
Let \( \theta_1, \theta_2, \theta_3 \) represent Euler angle: yaw, roll, and pitch, respectively. Then the coordinate transformation matrix can be expressed as follows:

\[
C^b_i = \begin{bmatrix}
\cos \theta_1 \cos \theta_2 - \sin \theta_2 \sin \theta_1 & \cos \theta_3 \sin \theta_1 + \sin \theta_3 \sin \theta_1 & -\cos \theta_3 \sin \theta_1 \\
-\cos \theta_2 \sin \theta_1 & \cos \theta_3 \cos \theta_1 & \sin \theta_3 \\
\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1 & \sin \theta_1 \sin \theta_2 - \sin \theta_2 \cos \theta_1 & \cos \theta_1 \cos \theta_3
\end{bmatrix}
\] (30)

Let \( \Delta \theta = [\Delta \theta_1 \Delta \theta_2 \Delta \theta_3]^T \) represent the Euler angle error, then the relationship between the error of coordinate transformation matrix and the Euler angle error can be expressed as

\[
\Delta C^b_i = L_1 \Delta \theta_1 + L_2 \Delta \theta_2 + L_3 \Delta \theta_3
\] (31)

where \( L_1, L_2 \) and \( L_3 \) are listed in Appendix A.

Write the last five items in Eq. (29) together as

\[
\gamma = \bar{C}_a^i \Delta C^b_i \lambda (v - \omega \times r) - \bar{C}_a^i \Delta C^b_i \bar{a}_s + \lambda \Delta v_a + \Delta a_s + \bar{C}_a^i \varepsilon
\] (32)

Then real measurement equation can be written as

\[
\bar{y} = -\bar{C}_a^i \bar{z} - \bar{a}_s
\]

\[
= -\frac{-\lambda (v_k - \omega_k \times r_k) + \gamma}{\bar{C}_a^i (v_k - \omega_k \times r_k) + \gamma}
\]

\[
= \frac{(v_k - \omega_k \times r_k)}{\|v_k - \omega_k \times r_k\|} \frac{\gamma}{\|v_k - \omega_k \times r_k\|} + \frac{\lambda (v_k - \omega_k \times r_k) \cdot \gamma}{\|v_k - \omega_k \times r_k\|} + \frac{\lambda (v_k - \omega_k \times r_k) \cdot \gamma}{\|v_k - \omega_k \times r_k\|} + \frac{\lambda (v_k - \omega_k \times r_k) \cdot \gamma}{\|v_k - \omega_k \times r_k\|}
\] (33)

where \( \bar{y} \) represents real normalized drag acceleration measurement. Separate the error part in Eq. (33) and rewrite it as \( \Delta y \), then the linear transfer formula from \( \Delta y \) to \( \gamma \) is as follows

\[
\Delta y = \left[ \begin{array}{c}
\frac{I}{\lambda \|v_k - \omega_k \times r_k\|} + \frac{(v_k - \omega_k \times r_k)(v_k - \omega_k \times r_k)^T}{\lambda \|v_k - \omega_k \times r_k\|^3}
\end{array} \right] \gamma
\] (34)

Considering that the variable \( \lambda \) cannot be calculated using the measurements, here divide \( \gamma \) into two parts:

\[
\gamma_1 = \bar{C}_a^i \Delta C^b_i \lambda (v - \omega \times r) + \lambda \Delta v_a \quad \text{which includes } \lambda , \quad \text{and } \gamma_2 = -\bar{C}_a^i \Delta C^b_i \bar{a}_s + \Delta a_s + \bar{C}_a^i \varepsilon \quad \text{which does not includes } \lambda .
\]

Then Eq. (34) can be divided into two parts to calculate as follows
\[
\Delta y_1 = \left[ -\frac{I}{\lambda \left( v_i - \omega_e \times r_k \right)} + \frac{(v_i - \omega_e \times r_k)(v_i - \omega_e \times r_k)^T}{\lambda \left( v_i - \omega_e \times r_k \right)} \right] \gamma_1
\]

\[
\Delta y_2 = \left[ -\frac{I}{\lambda \left( v_i - \omega_e \times r_k \right)} + \frac{(v_i - \omega_e \times r_k)(v_i - \omega_e \times r_k)^T}{\lambda \left( v_i - \omega_e \times r_k \right)} \right] \gamma_2
\]

\[
\Delta y = \Delta y_1 + \Delta y_2.
\]

So far, the covariance matrix of \( \Delta y \) can be completely expressed as

\[
R_y = H_{\theta \theta} \text{cov}(\Delta \theta) H_{\theta \theta}^T + H_\varepsilon \text{cov}(\varepsilon) H_\varepsilon^T + H_{a_\theta} \text{cov}(\Delta a_\theta) H_{a_\theta}^T + H_{a_\varepsilon} \text{cov}(\Delta a_\varepsilon) H_{a_\varepsilon}^T
\]

where \( H_{\theta \theta} , H_\varepsilon , H_{a_\theta} , \) and \( H_{a_\varepsilon} \) are partial derivative of \( \Delta y \) with respect to \( \Delta \theta , \varepsilon , \Delta a_\varepsilon \) and \( \Delta a_\theta \) respectively.

Then the covariance matrix of the initial orbit determination result error is

\[
P_x = \left( H^T R^{-1} H \right)^{-1}
\]

V. Simulation Results

A. Numerical Accuracy Verification

In order to verify the numerical accuracy of the proposed initial orbit determination method, three 24-hour Kepler orbits are simulated considering central gravitational field only. The orbital elements are listed in Table 2. The perigee altitudes of the three Kepler orbits are all set to 200 km and the apogee altitudes are 200 km, 300 km, and 400 km respectively. The non-conservative force acceleration data which are used for the initial orbit determination process
are generated by using NRLMSISE-00 and solar radiation pressure model with a time interval of 100 s. Errors are not considered in the generated non-conservative force data. The sampling time interval of the orbit determination method is set to 3600 s and the orbit determination process is conducted every 100 s. The error tolerance for position and velocity iteration is set to 1 m and 1 m/s. Under the conditions given above, the initial orbit determination is conducted 757 times in each case by using the proposed method and the results are shown in Table 3.

### Table 2 Orbital elements used for numerical accuracy verification

<table>
<thead>
<tr>
<th>Orbit number</th>
<th>Apogee altitude, km</th>
<th>Perigee altitude, km</th>
<th>Eccentricity</th>
<th>Inclination, °</th>
<th>Right ascension of ascending node, °</th>
<th>Argument of perigee, °</th>
<th>Mean anomaly, °</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>200</td>
<td>0</td>
<td>0.00754</td>
<td>60</td>
<td>120</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>300</td>
<td>200</td>
<td>0.0150</td>
<td>60</td>
<td>120</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>400</td>
<td>200</td>
<td>0.0150</td>
<td>60</td>
<td>120</td>
<td>0</td>
<td>30</td>
</tr>
</tbody>
</table>

### Table 3 Numerical accuracy results

<table>
<thead>
<tr>
<th>Orbit number</th>
<th>Apogee altitude, km</th>
<th>Perigee altitude, km</th>
<th>Initial value error</th>
<th>Convergence error</th>
<th>Average number of inner iteration</th>
<th>Average number of outer iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Position, m</td>
<td>Velocity, m/s</td>
<td>Position, m</td>
<td>Velocity, m/s</td>
</tr>
<tr>
<td>1</td>
<td>200</td>
<td>200</td>
<td>2.31×10^8</td>
<td>270.09</td>
<td>9.67×10^{-7}</td>
<td>4.67×10^{-10}</td>
</tr>
<tr>
<td>2</td>
<td>300</td>
<td>200</td>
<td>2.40×10^5</td>
<td>271.11</td>
<td>5.07×10^{-6}</td>
<td>4.02×10^{-9}</td>
</tr>
<tr>
<td>3</td>
<td>400</td>
<td>200</td>
<td>2.57×10^5</td>
<td>259.85</td>
<td>4.39×10^{-6}</td>
<td>2.97×10^{-9}</td>
</tr>
</tbody>
</table>

As is shown in Table 3, initial value errors of different orbits are all on the order of magnitude of 10^5 m and 10^2 m/s. These initial values calculated from the proposed initial value formulas can be used at the beginning of the numerical calculation because their errors are one order of magnitude lower than the absolute value of the position and velocity of the satellite. And after the implementation of initial orbit determination algorithm, the results converged and the errors are reduced to orders of magnitude of 10^{-6} m and 10^{-9} m/s, indicating good numerical accuracy. The average numbers of inner iteration and outer iteration are both very small which indicates that the initial values given by Eq. (16) and Eq. (18) are valid and the initial orbit determination algorithm has high efficiency.

### B. Influence of Different Sources of Error

In order to analyze the influence of different sources of error on orbit determination results, errors of different orders of magnitude which satisfy normal distribution are added to the generated non-conservative force acceleration measurements, solar radiation pressure acceleration, star sensor measurements and atmosphere rotational velocity. These data are generated with a time interval of 100 s. Three 24-hour Kepler orbits are generated considering central gravitational field only and the orbital elements are listed in Table 4. Then the initial orbit determination process is
performed every 100 s and the results are analyzed in detail. In the initial orbit determination process, different orbital altitudes and different sampling time intervals have influence on the orbit determination results. Thus they are also analyzed.

Table 4 Orbital elements used for initial orbit determination error analysis

<table>
<thead>
<tr>
<th>Orbit number</th>
<th>Perigee altitude, km</th>
<th>Eccentricity</th>
<th>Inclination, °</th>
<th>Right ascension of ascending node, °</th>
<th>Argument of perigee, °</th>
<th>Mean anomaly, °</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>200</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>300</td>
<td>0</td>
<td>60</td>
<td>120</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Firstly, the influence of accelerometer measurement noise is analyzed. Here Orbit 4 listed in Table 4 is adopted and the standard derivations of noise are set to $1 \times 10^{-9}$ m/s$^2$, $1 \times 10^{-10}$ m/s$^2$ and $1 \times 10^{-11}$ m/s$^2$, respectively. The other sources of error are not considered. The sampling time interval is set to 3600 s. The orbit determination results are shown in Figs. 3–5. In these figures, the black line represents the norms of actual position and velocity errors, and the red line represents the $3\sigma$ boundary of orbit determination errors obtained from covariance analysis, Eq. (39). It can be seen from Figs. 3–5 that the errors of orbit determination results increase with the increasing of noise standard derivation. Orbit 5 and Orbit 6 are also adopted in this part of calculation, the mean error and mean variance (1σ) are listed in Table 5. It shows that the errors of orbit determination results increase with the increasing of perigee altitude, which can be explained by the fact that the earth atmosphere density decreases rapidly as the altitude increases.

![Fig. 3 Error norms of initial orbit determination results ($\sigma_e$ set to $1 \times 10^{-9}$ m/s$^2$).](image1)

![Fig. 4 Error norms of initial orbit determination results ($\sigma_e$ set to $1 \times 10^{-10}$ m/s$^2$).](image2)
Secondly, the influence of solar radiation pressure model error on the orbit determination result is analyzed. The solar radiation pressure analysis results in Darugna et al. [3] have shown an uncertainty level on the order of $10^{-9}$ m/s$^2$ for the box-wing model and the ray-tracing model. Hence the standard derivations of modeling error are set to $1\times 10^{-8}$ m/s$^2$ and $1\times 10^{-9}$ m/s$^2$, respectively. Here Orbit 4 listed in Table 4 is adopted and the other sources of error are not considered. Sampling time interval is set to 3600 s. Position error norms of initial orbit determination results are shown in Fig. 6. Mean error and mean variance $(1\sigma)$ are listed in Table 6. Similar to the influence of accelerometer measurement noise, the errors of orbit determination results increase with the increasing of standard deviation of solar radiation pressure modeling error.
\[ \sigma_s = 1 \times 10^{-8} \text{ m/s}^2 \]
\[ \sigma_s = 1 \times 10^{-9} \text{ m/s}^2 \]

Fig. 6 Position error norms due to different solar radiation pressure model errors.

Table 6 Mean errors and variances for different standard derivations of solar radiation pressure model errors \((h_p=200 \text{ km})\)

<table>
<thead>
<tr>
<th>Error standard derivation, m/s²</th>
<th>Mean error Position, m</th>
<th>Velocity, m/s</th>
<th>Mean variance ((1\sigma)) Position, m</th>
<th>Velocity, m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 \times 10^{-8})</td>
<td>392.53</td>
<td>0.37</td>
<td>329.25</td>
<td>0.31</td>
</tr>
<tr>
<td>(1 \times 10^{-9})</td>
<td>39.26</td>
<td>0.037</td>
<td>32.93</td>
<td>0.031</td>
</tr>
</tbody>
</table>

Then the influence of star sensor noise on the orbit determination result is analyzed. Here Orbit 4 listed in Table 4 is adopted and the standard derivations of noise are set to 10 arcsec, 1 arcsec and 0.1 arcsec, respectively. The 10 arcsec and 1 arcsec noise levels represent the precision of most common star sensors based on Charge Coupled Device (CCD) or Complimentary Metal-Oxide Semiconductor (CMOS) image sensors and the 0.1 arcsec noise level represents the precision of the most advanced star sensors of nowadays. The other sources of error are not considered. Sampling time interval is set to 3600 s. Position error norms of initial orbit determination results are shown in Fig. 7. Mean error and mean variance \((1\sigma)\) are listed in Table 7. It can be seen that the errors of orbit determination results increase with the increasing of star sensor noise standard derivation.

Fig. 7 Position error norms due to different star sensor noises.
Table 7 Mean errors and variances for different star sensor noise standard derivations ($h_p=200$ km)

<table>
<thead>
<tr>
<th>Noise standard derivation, arcsec</th>
<th>Mean error</th>
<th>Mean variance (1σ)</th>
<th>Position, m</th>
<th>Velocity, m/s</th>
<th>Position, m</th>
<th>Velocity, m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>467.57</td>
<td>0.43</td>
<td>397.12</td>
<td>0.36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>47.70</td>
<td>0.045</td>
<td>39.71</td>
<td>0.036</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>4.97</td>
<td>0.0047</td>
<td>3.97</td>
<td>0.0036</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Next, the influence of earth atmosphere rotational velocity error on the orbit determination result is analyzed. The uncertainty level of the current most accurate Earth’s wind model HWM07 is on the order of 10 m/s [1]. Hence the standard derivations of velocity error are set to 15 m/s, 10 m/s and 5 m/s, respectively. The Orbit 4 listed in Table 4 is adopted. The other sources of error are not considered. Sampling time interval is set to 3600 s. Position error norms of initial orbit determination results are shown in Fig. 8. Mean error and mean variance (1σ) are listed in Table 8. It can be seen that the errors of orbit determination results increase with the increasing of atmosphere rotational velocity error standard derivation. Table 8 shows that the atmosphere rotational velocity error has a greater influence on the initial orbit determination result. The initial orbit determination error is on the order of several kilometers.

![Fig. 8 Position error norms of initial orbit determination results.](image)

Table 8 Mean errors and variances for different standard derivations of earth atmosphere rotational velocity errors ($h_p=200$ km)

<table>
<thead>
<tr>
<th>Error standard derivation, m/s</th>
<th>Mean error</th>
<th>Mean variance (1σ)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Position, m</td>
<td>Velocity, m/s</td>
</tr>
<tr>
<td>15</td>
<td>21129.07</td>
<td>19.95</td>
</tr>
<tr>
<td>10</td>
<td>14526.24</td>
<td>13.67</td>
</tr>
<tr>
<td>5</td>
<td>7413.40</td>
<td>7.01</td>
</tr>
</tbody>
</table>

Then the influence of sampling time interval on the orbit determination result is analyzed. Here Orbit 4 listed in Table 4 is adopted and the sampling time interval are set to 3600 s, 1800 s and 900 s, respectively. The standard derivations of accelerometer measurement noise, solar radiation pressure acceleration error, star sensor measurement noise, and atmosphere rotational velocity error are set to $1\times10^{-10}$ m/s$^2$, $1\times10^{-9}$ m/s$^2$, 0.1 arcsec and 5 m/s, respectively.
Mean error and mean variance (1σ) of initial orbit determination results are listed in Table 9. It can be seen from Table 9 that the errors of orbit determination results increase with the decreasing of sampling time interval and the proposed initial orbit determination method can obtain a good orbit determination result on the order of several kilometers under the circumstance which contains various types of errors.

Table 9 Mean errors and variances for different standard derivations of earth atmosphere rotational velocity errors (\(h_p = 200\) km)

<table>
<thead>
<tr>
<th>Sampling time interval, s</th>
<th>Mean error</th>
<th>Mean variance (1σ)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Position, m</td>
<td>Velocity, m/s</td>
</tr>
<tr>
<td>3600</td>
<td>7406.50</td>
<td>7.08</td>
</tr>
<tr>
<td>1800</td>
<td>7570.77</td>
<td>7.06</td>
</tr>
<tr>
<td>900</td>
<td>19128.78</td>
<td>9.14</td>
</tr>
</tbody>
</table>

Finally, the real accelerometer data from GOCE is used to test the proposed method. The data of GOCE’s early free reentry phase is from 23 October, 2013 to 30 October, 2013. In this phase, the orbital altitude of GOCE is about 225km and the accelerometers are not saturated. The acceleration measurements of GOCE are obtained from six tri-axial accelerometer in the form of common-mode measurements.

The accelerometer data of 24 October, 2013 is used for the verification of the proposed initial orbit determination method. The precise orbits obtained from GPS are used to estimate the scale factor and bias, which are then used to calibrate the raw measurements. In addition, the outliers are also removed. The accurate attitude quaternions obtained from star sensor are also used as inputs. The initial orbit determination results are compared with precise orbits as Fig. 9 shows. The mean error norm of initial orbit determination of GOCE is 4.86×10^4 m and 47.66 m/s.

![Fig. 9 Position and velocity error norms of GOCE initial orbit determination results.](image)

VI. Conclusion

In this Note, a novel initial orbit determination method utilizing the measurements of spaceborne accelerometer is proposed. Position accuracies of several kilometers are achieved under the conditions of the state-of-the-art accuracy.
levels of current accelerometer, star sensor, solar radiation pressure model and atmosphere rotational velocity model. Among all kinds of sources of error, the earth atmosphere rotational velocity error has the most significant impact on the initial orbit determination results. Position accuracy of tens of kilometers is achieved utilizing real accelerometer data of GOCE, revealing the possible practical use of this method in real missions. Since the superconducting/electrostatic accelerometers rely on vibration isolation platforms and independent thermal control system, the method might not be applicable for small satellites. However, the proposed initial orbit determination method does not rely on any other satellites or ground stations. It provides an ideal choice for fully autonomous navigation of low Earth orbiting satellites and deep space probes orbiting around planetary bodies with neutral atmosphere.

Appendix A: Partial Derivatives of Coordinate Transformation Matrix with Respect to Euler Angles

The partial derivatives of the coordinate transformation matrix from the ECI frame to satellite body frame with respect to the Euler angles \( \theta_1, \theta_2, \theta_3 \) are shown as follows.

\[
L_1 = \frac{\partial C^L_i}{\partial \theta_1} = \begin{bmatrix}
-\cos \theta_1 \sin \theta_1 & -\cos \theta_1 \cos \theta_1 & \cos \theta_1 \\
-\sin \theta_2 \cos \theta_1 & -\cos \theta_2 \cos \theta_1 & -\cos \theta_2 \\
-\sin \theta_3 \cos \theta_1 & -\cos \theta_3 \cos \theta_1 & 0
\end{bmatrix}
\]

(40)

\[
L_2 = \frac{\partial C^L_i}{\partial \theta_2} = \begin{bmatrix}
-\cos \theta_2 \sin \theta_2 \sin \theta_1 & -\cos \theta_2 \sin \theta_2 \cos \theta_1 & \cos \theta_2 \sin \theta_2 \\
\sin \theta_2 \cos \theta_2 \sin \theta_1 & \sin \theta_2 \cos \theta_2 \cos \theta_1 & -\sin \theta_2 \\
-\cos \theta_2 \cos \theta_2 \sin \theta_1 & -\cos \theta_2 \cos \theta_2 \cos \theta_1 & 0
\end{bmatrix}
\]

(41)

\[
L_3 = \frac{\partial C^L_i}{\partial \theta_3} = \begin{bmatrix}
-\sin \theta_3 \sin \theta_3 \sin \theta_1 & -\sin \theta_3 \sin \theta_3 \cos \theta_1 & \sin \theta_3 \sin \theta_3 \\
\sin \theta_3 \cos \theta_3 \sin \theta_1 & \sin \theta_3 \cos \theta_3 \cos \theta_1 & -\sin \theta_3 \\
-\cos \theta_3 \cos \theta_3 \sin \theta_1 & -\cos \theta_3 \cos \theta_3 \cos \theta_1 & 0
\end{bmatrix}
\]

(42)

Appendix B: Partial Derivatives of \( \Delta y \)

The partial derivatives of \( \Delta y \) with respect to \( \varepsilon, \Delta a, \Delta v, \Delta \theta \) respectively are shown as follows.

\[
H_\varepsilon = \left[ -\frac{I}{|C^z_i \cdot \bar{a}|} \frac{(v_i - \omega \times r_i) (v_i - \omega \times r_i)^T}{|C^z_i \cdot \bar{a}|^2} \right] C^z_i
\]

(43)

\[
H_{\Delta a} = \left[ -\frac{I}{|C^z_i \cdot \bar{a}|} \frac{(v_i - \omega \times r_i) (v_i - \omega \times r_i)^T}{|C^z_i \cdot \bar{a}|^2} \right] C^z_i
\]

(44)
\[
H_{s_k} = \left[ -\frac{I}{(v_k - \omega_E \times r_k)} + \frac{(v_k - \omega_E \times r_k)(v_k - \omega_E \times r_k)^T}{\|v_k - \omega_E \times r_k\|^3} \right]
\]

\[
H_{s_0} = \left[ -\frac{I}{(v_k - \omega_E \times r_k)} + \frac{(v_k - \omega_E \times r_k)(v_k - \omega_E \times r_k)^T}{\|v_k - \omega_E \times r_k\|^3} \right]
\]

\[
\frac{I}{C_i^i z - \bar{a}} - \frac{(v_k - \omega_E \times r_k)(v_k - \omega_E \times r_k)^T}{\|v_k - \omega_E \times r_k\|^2} \hat{C}^i_j L_j \bar{a}_j
\]

\[
\frac{I}{C_i^i z - \bar{a}} - \frac{(v_k - \omega_E \times r_k)(v_k - \omega_E \times r_k)^T}{\|v_k - \omega_E \times r_k\|^2} \hat{C}^i_j L_j \bar{a}_j
\]

\[
= \left[ \frac{I}{C_i^i z - \bar{a}} - \frac{(v_k - \omega_E \times r_k)(v_k - \omega_E \times r_k)^T}{\|v_k - \omega_E \times r_k\|^2} \hat{C}^i_j L_j \bar{a}_j \right]
\]

\[
= \left[ \frac{I}{C_i^i z - \bar{a}} - \frac{(v_k - \omega_E \times r_k)(v_k - \omega_E \times r_k)^T}{\|v_k - \omega_E \times r_k\|^2} \hat{C}^i_j L_j \bar{a}_j \right]
\]

\[
(45)
\]

\[
(46)
\]

**Funding Sources**

This research was supported by the Foundation of Key Laboratory of National Defense Science and Technology (No. 61422110303) and the Fundamental Research Funds for the Central Universities (No. YWF-19-BJ-J-276).

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